

# Stochastic Quantum Trajectories without a Wave Function

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## Abstract

After summarizing three versions of trajectory-based quantum mechanics, it is argued that only the original formulation due to Bohm, which uses the Schrödinger wave function to guide the particles, can be readily extended to particles with spin. To extend the two wave function-free formulations, it is argued that necessarily particle trajectories not only determine location, but also spin. Since spin values are discrete, it is natural to revert to a variation of Bohm's pilot wave formulation due originally to Bell. It is shown that within this formulation with stochastic quantum trajectories, a wave function free formulation can be obtained.

## 1 Introduction

In the last ten years or so, there has been a growing interest in Bohm's trajectory based interpretation of quantum mechanics [1, 2]. Even though the introduction of classical-like particle trajectories into the quantum mechanical state description does not have any observable consequence, it offers more practical ways to depict and explore quantum behavior. Specifically, in quantum chemistry this has led to novel

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ways to numerically solve and investigate multi-particle quantum systems [2]. From a conceptual or ontological point of view, Bohm's interpretation has a number of well-known advantages; the most compelling feature being the resolution of the measurement problem: unlike the traditional Copenhagen interpretation, there is no need for a collapse of the wave function. Obviously Bohm's trajectory based interpretation is not generally adopted and one can raise various objections and queries. For example: Why is there still a wave function in this picture - why should the particles need to be guided by a pilot wave? Why would there only be trajectories for position (or for fermion number, if we adopt Bell's version of Bohm's interpretation [3])? The experimental verification of Bell's inequalities [4] has effectively excluded local hidden variable interpretations of quantum mechanics. The presence of the pilot wave and its manifestation as a quantum potential are presumed to provide the required non-locality that allows compliance to Bell's inequalities, but how does this come about?

The first question has been addressed by Holland, Poirier and Hall et al. in recent work [6, 5, 7]. Interestingly, these authors show that Bohm's quantum trajectories can be obtained without a guiding wave function from a reformulated theory that prescribes the equations of motion for the particles, along with a probability distribution for the resulting particle trajectories. This probability distribution (or, in the version of ref. [7], the repulsion among trajectory realizations) generates an additional force that causes all required quantum effects.

The second question was addressed in [8], where it was shown that it is in fact possible to compute trajectories for all observables, including inherently discrete entities like spin. In this work it was argued that one could choose a preferred complete set of (commuting) observables, or - lacking compelling arguments to prefer one set over another - allow "all" observables, also if they are mutually non-commuting, to have well-defined trajectories.

The third question of how non-locality manifests itself in Bohm's quantum trajectory interpretation, has been discussed in detail in ref. [1]. Here it is shown how coherence in the multi-particle wave function acts as a non-local guidance for the particle trajectories and thus manages to avoid the constraints imposed by Bell's inequalities. Note that the pilot wave, which carries the entangled spin state, is essential to make the particles behave such that results of normal quantum mechanics are reproduced.

This paper revisits the above three points to further clarify the issues and show their interconnectedness. First, in section 2 three available formulations of Bohm’s trajectory interpretation of quantum mechanics for spin-zero particles are summarized. Since all three are designed to reproduce observable results of normal quantum mechanics, their difference is primarily in the underlying ontology, or in what are assumed to be “elements of reality”. Additional differences, for example how amenable each formulation is to numerical evaluation, will not be pursued in any detail here. The first formulation, supported by Bohm [1], Bell [3] and others [9] firmly assumes a pilot-wave, which is a solution of the Schrödinger equation, as part of reality; the two other formulations make no explicit reference to a wave function or Schrödinger equation [6, 5, 7].

Then, in section 3, it will be argued that such wave function-free formulations of quantum mechanics that only give a “beable” status to position, when carried over to particles with spin, will lack the state information that is required to provide the quantum correlations required to describe particles with spin. This requires that also spin state has to be included somehow as an element of reality, as is the case for the stochastic trajectory formulation of quantum mechanics developed in ref. [8]. Hence, it suggests to explore if the wave function-free formulations discussed in section 2 can be extended to this stochastic trajectory formulation. This is carried out in section 4, where it is shown that the ensemble of quantum trajectories for any (discrete) observable can be generated self consistently from a suitably chosen distribution of initial values by applying a local (stochastic) evolution rule without reference to a wave function. As in the wave function-free particle formulation, the dynamics of the stochastically evolving quantum numbers is affected by the local potential and non-local probability distribution defined by the ensemble of trajectories in the high-dimensional state space. Finally, some concluding comments are presented in Section 6.

## 2 Bohm Trajectories without Wave function: Three formulations

To keep notations simple and following refs. [5, 7], this section will consider quantum mechanics of a single particle without spin in one dimension. It is mostly straightforward to extend the discussion to

non-relativistic quantum mechanics for multiple spin zero particles in three space dimensions. The Schrödinger equation for such a one-particle system is given by

$$i\hbar\dot{\psi}(x,t) = -\frac{\hbar^2}{2m}\psi''(x,t) + V(x)\psi(x,t), \quad (1)$$

where  $m$  is the particle mass,  $V$  the classical potential and  $\hbar$  Planck's constant. As explained in e.g. [1], for a quantum system described by the wave function  $\psi(x,t)$  one can define an ensemble of particle trajectories  $x(t, x_0)$ . A family of trajectories can be derived from the wave function

$$\psi = Re^{iS/\hbar}. \quad (2)$$

using the “pilot” equation,

$$\dot{x}(t, x_0) = S'(x_0, t)/m, \quad (3)$$

where  $S'(x_0, t)$  is the gradient of the phase of the wave function, evaluated at  $x(t_0) = x_0$ . It follows from the Schrödinger equation that these particle trajectories obey almost classical equations of motion,

$$\ddot{x} = -(V(x) + Q(x, t))'/m. \quad (4)$$

Quantum effects are due to the quantum potential that is present in addition to the classical potential  $V$ ,

$$Q = \frac{-\hbar^2}{2m} \frac{R''}{R}. \quad (5)$$

Quantum uncertainty arises because one must consider the full ensemble of trajectories, with all possible initial conditions  $x_0$ , which must be distributed initially according to the probability density at starting time  $t_0$ ,  $P(x_0, t_0)$ . The probability density for particle positions is computed from the radial part of the wave function as,

$$P(x, t) = R^2(x, t). \quad (6)$$

When evaluating probabilities as averages over this evolving ensemble of particles, the same results as in normal quantum mechanics are recovered.

To solve the dynamics of this ensemble of particles, one can follow at least three approaches, which will be summarized next.

## 2.1 Trajectories Guided by a Wave Function (F-I)

The most straightforward (and oldest [1, 9]) approach is to first solve the (linear) Schrödinger equation to obtain the complex valued  $\psi(x, t)$ . After computing  $\psi(x, t)$ , one can use its phase  $S(x, t)$  with eq. (3) to compute particle trajectories. The radial part of the wave function does not play a direct role in this formulation, other than providing the probability distribution for the initial trajectory positions  $x_0$ .<sup>1</sup>

## 2.2 Trajectories with Self-contained Dynamics (F-II)

Second, one can exploit the particle aspect of the dynamics more explicitly and compute particle trajectories from the equation of motion (4) for the particles. Unfortunately, this equation contains the quantum potential, which is only given at the initial time and subsequently evolves dynamically as well. In contrast with the previous formulation F-I, the quantum potential in this formulation is computed as an emergent property of the ensemble of particles; The function  $R(x, t)$  needed to compute the quantum potential (5) is obtained as the square root of the probability density of particle positions in the evolving ensemble of particle trajectories.

Hence, one can apply the following leap-frog scheme for solving the particle dynamics: Choose a large set of initial positions, according to a (given) probability density,  $P(x_0, t_0)$ , along with initial values for the particle velocities,  $v(x_0, t_0)$ . Compute the quantum potential at this initial time, using (6) and (5). Make sure that the initial velocities satisfy suitable integrability conditions [11], such that they can be obtained as the gradient of an underlying scalar function,

$$v(x, t_0) = S'(x, t_0)/m. \quad (7)$$

Note that this is not straightforward, nor particularly natural, in more than one dimension. Then evolve the particle positions in the ensemble from  $t_0$  to  $t = t_0 + dt$ , using the almost classical equations of motion,

$$v(x, t + dt) = -\frac{1}{m}(V(x) - \frac{\hbar^2}{2m} \frac{(P^{1/2}(x, t))''}{P^{1/2}(x, t)})' dt, \quad (8)$$

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<sup>1</sup>It is even argued [1, 9] that the proper probability distribution,  $P(x, t) = R^2(x, t)$  will develop dynamically from an arbitrary initial distribution, since only this specific distribution is conserved during the non-linear dynamics of the particles.

$$x(x, t + dt) = x(x, t) + (v(x, t + dt))dt. \quad (9)$$

Here, the quantum potential is computed from the probability density for particle positions, using eq. (6). Notice that the velocity is updated by adding a gradient, hence condition (7) remains valid also for later times. After propagating all particle positions in the ensemble to the incremented time, the new probability density can be computed using a suitable histogramming or averaging technique,

$$P(x, t + dt) \propto (\#\text{particles} \in [x, x + a])/a. \quad (10)$$

It can be shown [6, 5, 7, 12] that this formulation generates fields  $R(x, t)$  and  $S(x, t)$ , which can be computed from  $P(x, t)$  and  $v(x, t)$ , and which can be combined into a wave function  $\psi(x, t) = R(x, t)e^{iS(x, t)/\hbar}$  that solves the Schrödinger equation (1). Hence, as in F-I, this formulation produces the same observable results as normal quantum mechanics. This is the case when an infinitely large ensemble of trajectories is used to compute the probability density (10). When a finite ensemble is used to estimate the distribution, as proposed in [7], the quantum potential featuring in (8) will be an approximation as well, and deviations from normal quantum mechanics will be implied.

Even though this scheme looks attractive from a numerical point of view, because it is relatively simple to solve the time evolution of the particles, the obvious drawback is the difficulty in computing the time evolution of the quantum potential (or, equivalently, the probability density of particle positions). One would need either clever approximation methods, or a very large ensemble of particle trajectories - assuming that only very small observable deviations from normal quantum mechanics can be allowed. In particular when the number of particles increases, the high dimensionality of the state space over which the probability density must be evaluated will require a very large ensemble size that will make any sampling method quickly prohibitively expensive.

## 2.3 Trajectories with Self-consistent Ensemble Dynamics (F-III)

The third formulation pays tribute to the quantum fluid approach originally introduced by Madelung [10] and more recently picked up by the quantum chemistry community [2, 5]. It uses the underlying particle dynamics, as in section 2.2 above, but use a different method

to compute the evolving quantum potential. The formulation is best explained in discretized form, where the underlying space is replaced by a grid. In one dimension, the particle positions are then given by

$$x_k = ka, \quad (11)$$

where  $a$  denotes the grid spacing, which for simplicity is taken to be constant throughout space and  $k$  is the grid cell index. Similarly as before, the initial state of the system is defined by specifying for all  $k$ , the probability density  $P_k(t_0)$  for the particle to be located in cell  $k$ , along with the initial value of the velocity  $v_k(t_0)$  that the particle would have in this cell. Hence, the state of the single particle is fully defined by the values of  $P_k$  and  $v_k$  for all  $k$ .

As in formulation F-II, the initial values of the velocities in each cell must be chosen subject to (the discretized version of) the constraint (7), and the velocities are updated using the almost classical equations of motion (8). The probability density is updated by enforcing the continuity equation,

$$P_k(t + dt) = P_k(t) + (P_{k-1}(t)v_{k-1}(t) - P_k(t)v_k(t))dt/a, \quad (12)$$

where it is assumed that the particles can only move to their direct neighboring cell, and that the velocity in cell  $k$  is in the positive direction.

In this “quantum hydrodynamics” formulation as it is presented above, there is no need to first solve the Schrödinger equation, in order to compute the ensemble of particle trajectories. Hence, also this third scheme describes particle dynamics without a guiding wave function. However, it is also clear that the computational effort to simultaneously solve  $P_k$  and  $v_k$  for all  $k$  could be *grosso-modo* the same as solving the the Schrödinger equation for  $R$  and  $S$ .

## 2.4 Similarities and Differences

Before closing this section, it may be worthwhile to summarize similarities and differences between the three formulations above. It is assumed here that the ensemble in formulation F-II (section 2.2) is sufficiently large, such that the probability density can be accurately computed, and that the grid spacing  $a$  in formulation F-III (section 2.3) is sufficiently small to avoid discretization effects. For a given initial state, all three methods then produce the same trajectories for

an ensemble of particles and reproduce all observable results of normal quantum mechanics. Also, as required [4]), all formulations have non-locality built into their dynamics: in all formulations, particle trajectories are influenced by the same quantum potential, which is an inseparable function over the multi-particle state space and hence intrinsically non-local.

There will, of course, be differences in computational strategies that are most natural or most efficient when implementing a numerical scheme for each of these formulations. However, this is not further pursued here. There are also important differences in the ontology implied by each approach, which will be outlined next.

In the traditional Bohm interpretation F-I, the wave function along with a particle trajectory, are elements of reality. The express purpose of the wave function is to guide the particle along its quantum trajectory. Quantum effects, such as interference effects but also entanglement of (spin) states and non-locality, are carried over from the wave function, which is evaluated over full state space, to the quantum potential [1] or transition probabilities [3],

In formulation F-II, there is a very large, possibly infinite, ensemble of particle trajectories, but no wave function. Particles move according to their equations of motion, as in classical dynamics, but the equation of motion are endowed with an extra potential (or force) term. This (again inseparable) quantum potential is generated as a collective effect of the entire ensemble of coexisting state space trajectories. Hence, all trajectories are equally important elements of reality.

For formulation F-III one can adopt an ontology that is close to either of the two above, depending on how the ensemble of trajectories is interpreted. One can represent the ensemble using the probability density and velocity field and give these a similar status as (the modulus and phase of) the wave function in the Bohm interpretation F-I. These evolving probability and velocity fields produce a quantum potential that, along with the classical potential, determines the (single) trajectory for the particle. Hence, almost as in F-I, the particle trajectory along with the probability density and velocity field are elements of reality. This flavor will be referred to as F-IIIa. Alternatively, one can represent the ensemble using a (very large but not necessarily infinite) ensemble of trajectories and effectively adopt the same ontology or elements of reality as in F-II. This alternative will be referred to as F-IIIb.



### 3 Particles with Spin

Until now, the particles were simple point-particles, with only ‘position’ as intrinsic property that is an element of reality. One can argue [1], that it is in fact sufficient that a particle *only* has a well-defined position. Other observable properties, such as momentum or spin, need not be intrinsic properties of the particle (and hence elements of reality) since they are only indirectly observed through various types of measurements that in the end boil down to registrations of pointer positions or other (displayed or printed) position-type indicators. Even though a well-defined spin is not an actual or intrinsic property of the moving particle, its trajectory can still be influenced by spin, because the guiding wave function evolves differently depending on its spin attributes, which are driven by spin-dependent terms in the Hamiltonian. In this way the results of the measurement process involving particles spin, including particles with entangled states, can reproduce the normal results of quantum mechanics [1][ch.10,12],[3][Ch.18].

Since wave function-free formulations require that the dynamics of the evolving state can be computed self-consistently without recourse to a guiding wave function, this seems to preclude extending formulations F-II and F-III to quantum mechanics for particles with spin, unless the particles are also endowed with spin degrees of freedom. As Bohm noted [1][Ch.10.4], this appears to be difficult, since spin degrees of freedom linked to a particle would grow proportionally with the number of particles, whereas the actual number of spin degrees of freedom grows exponentially. Hence, Bohm conceded that for particles with spin, a formulation in which trajectories are determined by the influence of a combination of classical and quantum forces could no longer be maintained. If it is possible to avoid Bohm’s conclusion and find a wave function-free formulation for trajectories for particles with spin, it appears inevitable to start from an ontology that includes full multi-particle spin state as an element of reality. Such a formulation was presented in ref. [8], which showed how “de Broglie-Bohm-Bell” (BBB) trajectories could be computed for any observable - also for multi-particle systems with both position and spin. The next section will show how this approach can be reformulated to provide a wave function-free description of the dynamics.

## 4 BBB Trajectories without Wave Function

Following ref. [8], all observables are assumed to take discrete values in this section. The Schrödinger wave equation for any discrete quantum system, including multi-particle systems with spin can be written as

$$i\hbar\dot{\psi}_n = \sum_m H_{nm}\psi_m, \quad (13)$$

where the composite index  $n$  labels the discrete values of the relevant observables. For example, eq. (13) can represent the Pauli equation for two-component wave functions and particles moving on a line or circle. Then  $n \equiv k, s$ , with  $k = x_k/a$  labeling position (cf. (11)), and  $s$  labeling the spin state.

Stochastic trajectories for the discrete values of the system state  $n$  can be computed as follows. In a small time interval  $dt$ , state  $m$  will jump to  $n$  with probability  $P(m \rightarrow n)$ , where

$$P(m \rightarrow n) = T_{nm}dt. \quad (14)$$

The transition probability  $T_{nm}$  is defined as

$$T_{nm} = \max(0, J_{nm}/P_m), \quad (15)$$

where  $J$  is the probability current,

$$J_{nm} = \frac{2}{\hbar} \text{Im}(\psi_n^* H_{nm} \psi_m), \quad (16)$$

which is anti-symmetric in  $n$  and  $m$ , and  $P_m$  is the usual probability for state  $m$ ,

$$P_m = \psi_m^* \psi_m. \quad (17)$$

The matrix  $H_{mn}$  is the Hamiltonian in the  $n$  representation. Further details are in refs. [8] and [3][Ch.19]. The above prescription for  $n$ -trajectories has the form of a guidance relation similar to (3) for (continuous) particle position: to compute the transition probabilities for time  $t$ , the wave function  $\psi_n$  at time  $t$  must be computed from the wave equation (13).

To obtain a wave function-free dynamics, the time-dependence of  $T_{nm}$  should be computed without recourse to a guiding wave function. As in formulation F-III, the initial state should then be defined as an

ensemble of  $n$ -values, with prescribed probabilities  $P_n$ . The stochastic trajectories along with the evolving probabilities  $P_n(t)$  can then be computed as,

$$\dot{P}_n(t) = P_n(t) \sum_m (T_{nm}(t)P_m(t) - T_{mn}(t)P_m(t)) \quad (18)$$

provided that the time-dependence of  $T_{nm}$  can be computed from a relation like

$$\dot{T}_{nm}(t) = F(T, P). \quad (19)$$

The function  $F$  may depend on the transition rates  $T$  and probabilities  $P$ , but must not refer to a pre-computed wave function as in (15,16).

To discover the evolution equation (19), it is convenient to first find an evolution equation for the currents  $J_{nm}$ , and then use (15) to convert it into an evolution equation for the transition rates  $T_{nm}$ . The first step then is to rewrite the probability current(16) using a wave function in polar form (2),

$$J_{nm} = -\frac{i}{\hbar}(R_n H_{nm} R_m e^{-i(S_n - S_m)/\hbar} - R_m H_{mn} R_n e^{i(S_n - S_m)/\hbar}) \quad (20)$$

and use this relation to solve for the wave function phase, or rather for

$$\theta_{nm} = e^{-i(S_n - S_m)/\hbar}. \quad (21)$$

Since (20) leads to a quadratic equation for  $\theta_{nm}$ , there will be two solutions for a given  $J_{nm}$ . However, consistency requires that  $\theta_{nm} = 1$  when  $S_n = S_m$ , which implies that the solution takes the form,

$$\theta_{nm} = \frac{[H_{nm}]\alpha_{nm} + i\hbar J_{nm}}{2R_m R_n H_{nm}}, \quad (22)$$

where

$$[H_{nm}] = +\text{Sign}(\text{Re}(H_{nm}))|H_{nm}| \quad (23)$$

and

$$\alpha_{nm} = (4R_m^2 R_n^2 - \frac{\hbar^2 J_{nm}^2}{|H_{nm}|^2})^{1/2}. \quad (24)$$

It can be readily shown that the term in the square root is non-negative, hence  $\alpha$  is real valued, with  $\alpha_{nm} = \alpha_{mn} \geq 0$ . The definition (23) shows how the sign ambiguity is fixed; note that  $[H_{mn}] = [H_{nm}]$  and  $[H_{nm}] = H_{nm}$  for a real valued Hamiltonian matrix.

As stated above, a wave function-free dynamics, requires an expression for  $\dot{J}$ , which may depend on  $J$ ,  $R$  (or  $P$ ) and the Hamiltonian, but

not on the phases  $S$ . In such a formulation, the phases  $S$  (or equivalently, the  $\theta$ ) are derived properties, which can be back-computed from the evolving values of  $J$  and  $R$ . It should be noted that the initial values for  $T$ , or equivalently  $J$ , cannot be chosen freely, since  $J_{nm}$  must obey (20) in which the  $R_n$  and  $S_n$  are the independent degrees of freedom<sup>2</sup>.

The sign-choice implemented above assumes that in the initial state the real part of  $\theta_{nm}$  is larger than zero. If, during subsequent time evolution

$$\alpha_{nm} \rightarrow 0, \quad (25)$$

it may be required to flip the sign in (23) such that  $\theta_{nm}$  has a continuous time dependence.

Using the result (22) the time-dependence of  $J_{nm}$  can be computed from

$$\dot{J}_{nm} = \frac{2}{\hbar^2} \text{Re} \left( \sum_{k|n} R_k H_{kn} H_{nm} R_m \theta_{mn}^* \theta_{nk}^* - \sum_{k|m} R_n H_{nm} H_{mk} R_k \theta_{nm} \theta_{mk} \right) \quad (26)$$

as

$$\dot{J}_{nm} = \frac{1}{2R_n^2} \left( J_{nm} \sum_{k|n} J_{nk} - \frac{1}{\hbar^2} [H_{nm}] \alpha_{nm} \sum_{k|n} [H_{nk}] \alpha_{nk} \right) - (m \leftrightarrow n). \quad (27)$$

The notation  $k|n$  in the summations indicates that only  $k$  values for which  $H_{kn} = H_{nk}^*$  is not equal to zero are included in the sum.

To turn eq. (27) into an evolution equation for the transition rates it is convenient to introduce an generalized transition rate  $\bar{T}$ , defined as

$$\bar{T}_{nm} = J_{nm}/P_m, \quad (28)$$

where  $\bar{T}_{nm}$  is allowed to be negative, and

$$\dot{\bar{T}}_{nm} = \dot{J}_{nm}/P_m - (J_{nm}/P_m) \dot{P}_m/P_m. \quad (29)$$

Using (27) and the continuity equation

$$\dot{P}_m = \sum_{k|m} J_{mk}, \quad (30)$$

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<sup>2</sup>It is assumed here that there are enough non-zero off-diagonal matrix elements in the Hamiltonian, such that for each independent  $S_n$  (except one, because the overall phase is irrelevant) there is at least one non-zero element  $H_{nm}$ , and hence  $J_{nm}$ .

the final result is obtained as,

$$\begin{aligned}\dot{\bar{T}}_{nm} = & -\frac{1}{2}\bar{T}_{nm}(\sum_{k|n}\bar{T}_{kn} - \sum_{k|m}\bar{T}_{km}) \\ & + \frac{1}{2\hbar^2}P_m^{-1}[H_{nm}]\alpha_{nm}(P_n^{-1}\sum_{k|n}[H_{kn}]\alpha_{kn} - P_m^{-1}\sum_{k|m}[H_{km}]\alpha_{km})\end{aligned}\quad (31)$$

With this result, a wave function-free formulation for the stochastic  $n$ -trajectories has been obtained, which is similar to formulation F-III. Specifically, the procedure to compute stochastic trajectories for the  $n$ -values is as follows. Define initial conditions for the trajectories of all  $n$ , by choosing probabilities  $P_n$  and transition rates  $T_{nm}$ . Since not all  $T_{nm}$  can be chosen independently, it is actually easier to choose initial values for the phase  $S_n$  and then use eqs. (20) – along with the chosen values for  $R_n = P_n^{1/2}$  – to compute the  $J_{mn}$  and from these the  $\bar{T}_{mn}$  and  $T_{mn}$ . In this way  $T_{mn}$  will satisfy all required consistency conditions and subsequent evolution using (32) will preserve these conditions. Finally, a sufficiently small time step size  $dt$  must be chosen, such that also  $T_{mm}dt = 1 - \sum_n T_{nm}dt > 0$  holds for all  $m$ . The time-dependence of the system is obtained by iteratively updating the probabilities  $P_n$  and transition rates  $T_{nm}$ , using

$$\begin{aligned}P_n(t+dt) &= P_n(t) + \sum_m (T_{nm}(t)P_m(t) - T_{mn}(t)P_n(t)), \\ \bar{T}_{nm}(t+dt) &= \bar{T}_{nm}(t) + \dot{\bar{T}}_{nm}(t)dt, \\ T_{nm}(t+dt) &= \max(0, \bar{T}_{nm}(t+dt)),\end{aligned}\quad (32)$$

where  $\dot{\bar{T}}$  is computed from eq. 31). The sign-choice implied in the  $[H]$ -terms in (31) should be the same as used for the previously time step, unless the cross-over condition (25) is encountered, which might required to flip the sign in (23).

## 4.1 Particle Moving on a Circle

The update algorithm (32) is conceptually similar to formulation F-III for spin-zero particles in continuous (1D) space. To show this more explicitly, it is instructive to compute the update rules in the continuum limit for a particle moving on a circle, for which the Hamiltonian can be written as (cf. [8]),

$$H_{mn} = (V_n + \frac{\hbar^2}{Ma^2})\delta_{m,n} - \frac{\hbar^2}{2Ma^2}(\delta_{m+1,n} + \delta_{m,n+1}), \quad (33)$$

with  $V_n$  the potential,  $M$  the particle mass and  $a$  the grid spacing. In this case, the evolution equations of formulation F-III should be recovered.

For a state in which the “particle” moves in the positive  $n$  direction, it is sufficient to compute the transitions rates from  $n$  to  $n + 1$ . This rate multiplied with the grid cell spacing  $a$ , can be interpreted as the average particle velocity,  $v_n$  (cf. (12)). Using this change in notation, the generalized transition rates can be written as

$$\bar{T}_{n+1,n} = \dot{v}_n/a, \quad \bar{T}_{n,n+1} = -v_n P_n/a P_{n+1}. \quad (34)$$

Using this definition of velocity, the evolution equation (31) turns into

$$\dot{v}_n = \frac{1}{2a} v_n (-v_{n-1} \frac{P_{n-1}}{P_n} + v_n + v_n \frac{P_n}{P_{n+1}} - v_{n+1}) + (\text{H-terms}), \quad (35)$$

where “H-terms” indicate the second group of terms in (31), which will be considered shortly. Since the particle moves in one space dimension, the summations in (31) only have two terms, which are explicitly shown in (35). To investigate the continuum limit,  $a \rightarrow 0$ , the velocities are expanded to order  $a$ ,  $v_{n\pm 1} = v_n \pm a v'_n$  and similarly for  $P_{n\pm 1}$ . Substitution in (35) shows that  $O(1)$  and  $O(a)$  terms cancel, such that only the H-terms remain.

Eq. (33) shows that  $H_{n+1,n} = O(1/a^2)$ , from which it follows that  $\alpha_{n+1,n} = 2R_{n+1}R_n + O(a^2)$  and hence,

$$\dot{v}_n = \frac{2a}{\hbar^2} \frac{[H_{n+1,n}]}{R_n^2} (R_n \sum_{k|n+1} R_k [H_{k,n+1}] - R_{n+1} \sum_{k|n} R_k [H_{k,n}]) + O(a). \quad (36)$$

Since the Hamiltonian matrix elements are real valued, the square brackets can be dropped and (33) implies that

$$[H_{n+1,n}] = [H_{n,n+1}] = -\hbar^2/2Ma^2, \quad [H_{n,n}] = V_n + \hbar^2/Ma^2. \quad (37)$$

After substituting this in (36) and expanding the shifted terms  $R_{n+k}$  to third order in  $a$  as  $R_{n+k} = R_n + akR'_n + a^2k^2R''_n/2 + a^3k^3R'''_n/6$ , one finds that  $O(1)$ ,  $O(a)$  and  $O(a^2)$  terms cancel and the expected result emerges,

$$\dot{v}_n = -\frac{1}{M} (V_n - \frac{\hbar^2 R''_n}{2MR_n})' + O(a). \quad (38)$$

## 4.2 Particle with Spin 1/2

To keep this example simple, the particle is assumed to have a very large mass such that its motion can be ignored, and to only interact with a magnetic field  $B$ . Hence, the  $2 \times 2$  Hamiltonian takes the simple form,

$$H = \mu \sigma \dot{\mathbf{B}} = \mu B \sigma_x, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (39)$$

where the  $x$  axis has been aligned with the magnetic field. Since  $n$  takes only two values, there is only one independent phase (the arbitrary global phase can be used to set one of the phases equal to 0) and one independent probability (since  $P_1 = 1 - P_2$ ). Therefore there is also only one transition rate  $\bar{T}_{12} = -\bar{T}_{21}P_1/P_2$  that has to be solved. Eq. (31) therefore simplifies to

$$\dot{\bar{T}}_{12} = \frac{1}{2}\bar{T}_{12}^2(1 + \frac{P_2}{P_1}) + \frac{\alpha_{12}^2[H_{12}]^2}{2\hbar^2 P_2}(\frac{1}{P_1} - \frac{1}{P_2}). \quad (40)$$

Using  $[H_{12}] = \mu B$  and

$$\alpha_{12} = (4P_1P_2 - \frac{\hbar^2}{\mu^2 B^2}\bar{T}_{12}P_2^2)^{1/2} \quad (41)$$

it follows that

$$\dot{\bar{T}}_{12} = \bar{T}_{12}^2 + \frac{2\mu^2 B^2}{\hbar^2}(2 - \frac{1}{P_2}). \quad (42)$$

Differentiating the continuity equation  $\dot{P}_2 = -\bar{T}_{12}P_2$  and eliminating  $\bar{T}_{12}$  using (42) gives

$$\ddot{P}_2 = -2\frac{2\mu^2 B^2}{\hbar^2}(P_2 - 1/2). \quad (43)$$

This equation is easily solved as

$$P_2(t) = (1 + \cos(2\gamma t + \delta))/2 = \cos^2(\gamma t + \delta), \quad (44)$$

with  $\gamma = 2\mu^2 B^2/\hbar^2$  and  $\delta$  a free integration constant. The generalized transition rate then follows as

$$\bar{T}_{12} = 2\gamma \tan(\gamma t + \delta). \quad (45)$$

It is straightforward to check that these results exactly agree with the results obtained from the wave function that solves the Schrödinger equation for this system.

## 5 Revisiting Entangled Spin States

With the wave function free formulation as developed above, it is clear that also quantum non-locality and entangled spin states can be properly captured. This is the case, because there is a one-to-one relation between the dynamics of the probability field  $P_n$  and (generalized) transition probabilities  $T_{nm}$  and the modulus  $R_n$  and wave function phases  $S_n$  - provided that the initial transition rates (or equivalently currents  $J_{nm}$  are chosen in accordance with the constraint (20). Hence, the description of the measurement process, including non-local effects due to entangled quantum states that was presented in [1][Ch.6,7] remains valid for the alternative formulation presented here. In formulation F-I the guiding wave function contains the extra spin state information, in the stochastic wave function-free dynamics, the state itself, which includes spin and position values as well as velocities and transition rates, contains all required information. Even though the evolution equations (32) may seem to be local, they of course describe the dynamics of a single point in the high-dimensional state space. An ensemble of such states can capture all required [4] correlations between the individual components of this state vector, also when these components refer to particle locations and spins at widely separated locations.

## 6 Discussion

The results of this paper extend previously developed formulations of quantum dynamics without wave functions to (non-relativistic) particles with spin. Specifically the formulation developed in section 3 shows that the stochastic trajectories of the alternative formulation of quantum mechanics in terms of discrete “beables” [8], can be generated without recourse to a guiding wave function. Exploring such an alternative might offer new insights. In particular it simplifies the “beable” world, or ontology, compared to a world in the Bohm-Bell formulation, F-I, which needs two different entities, particle locations with classical-like trajectories and a complex valued wave function that evolves according to the Schrödinger equation. Also, having an alternative set of evolution equations could lead to different, possibly more efficient numerical methods to solve and depict quantum dynamics.

However, the stochastic dynamics developed above also has less



attractive features. Since the dynamics is computed in terms of the over-complete set of generalized transition rates, instead of the wave function phases, the initial conditions are constrained in a rather complex and unnatural fashion. Something similar is present in the wave function-free formulations F-II and F-III: here the particle trajectories have a constraint on the initial velocities (two or more dimensions). Even formulation F-I has a constraint on the initial probability density for particle positions, which must be chosen to be equal to the modulus squared of the wave function. However, here Bohm and others [1, 9] argue that these constrained initial conditions may not be required, since deviations will dynamically average out. In the current formulation, which lacks a guiding wave function, such a dynamical recovery of the interdependency of probability current and wave function phase seems very unlikely.

Also the gain in simplicity compared to the Bohm-Bell formulation F-I is perhaps fairly meager: As in formulation F-III, one still needs to solve for the probabilities and transition rates in the full state space, i.e., an equally imposing task as solving the guiding wave function in F-I.

Finally, there is the question if it is natural to adopt an integer spin value as an element of reality. In itself this is attractive, since it avoids having to argue that some quantum state properties are fundamentally different from others. I.e., one needs not argue that “location” is an intrinsic property, also in the microscopic world, but “spin” is not, and – unlike position – only emerges when macroscopic measurements are performed. Of course, as discussed in ref. [8], if spin would be granted a “beable” status, it is still not clear *which* spin: In the formulation developed above, eigenvalues of any spin operator can be “beables”. Choosing for example eigenvalues of  $\sigma_z$ , the probabilities and transition rates are directly relevant for the evolution of the spin in this  $z$ -direction. Measurement result of  $\sigma_z$  can then be directly understood, i.e., “read off” from the quantum mechanical state – as is the case for properties in a classical state. However, results of measurements of other (non-commuting) spins can no longer be understood directly and one has to invoke the same arguments as used in F-I ([1][Ch.6,7] to understand how measurements of these spins will still agree with the results of normal quantum mechanics.

It may be interesting to note at this point that one could resort to the view proposed in ref. [8], that in fact *all* observables have a “beable” status. For spin this would imply that the particle has many

coexisting spin properties, for all possible spin operators (i.e., a dense, but still finite subset of all possible Hermitian and unitary  $2 \times 2$  matrices). Quantum entanglement then plays out in a perhaps surprising, but very compelling way: In the typical set-up of two free moving particles with entangled spins, the spin part of the Hamiltonian is zero, and hence the spin state remains unchanged while the particles move away from each other. Therefore it may seem that this system has the same (non)locality as a classical system in which some properties are correlated over long distance, simply because they started out that way. In order then to understand how the correct quantum results are obtained when performing spin measurements in arbitrary relative directions – without evoking the decohering effect of a measuring apparatus - one must assume that the specific set-up of the measuring apparatus, specifically reads-off the property from the trajectory with a spin direction that aligns with the device. However, this cannot be done in a local fashion: the specific two particles spin realization is picked-up for which each of the *two* spin operators aligns with the corresponding measuring device. From a quantum trajectory point of view, the combined measuring apparatus reads-off a specific state from one trajectory, but this trajectory of course describes the correlated spin values for two widely separated particles.

On the one hand, this coexisting states interpretation of quantum mechanics compellingly shows the non-local nature of “The Undivided Universe”; on the other hand – to quote Bell – “It seems an eccentric way to make a world”.

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